Abelian Functions and Singularity Theory

Victor Buchstaber
Steklov Institute, RAS
⟨buchstab@mi.ras.ru⟩

Based on the results of joint work with

Dmitry Leykin
Institute of magnetism, Kiev
⟨dile@imag.kiev.ua⟩

Moscow
August 2007

Abstract

In the cycle of our joint papers with V.Enolski and D.Leykin we have developed a theory of multivariate sigma-function, an analogue of the classic Weierstrass sigma-function.

Sigma-function is defined on a covering of U, where U is the space of a bundle $p : U \to B$ defined by a family of plane algebraic curves of fixed genus. The base B of the bundle is the space of the family parameters and a fiber J_b over $b \in B$ is the Jacobi variety of the curve with the parameters b. A second logarithmic derivative of the sigma-function along the fiber is an Abelian function on J_b.

Using sigma-function, one can generate a ring F of fiber-wise Abelian functions on U. The problem to find derivations of the ring F along the base B is a reformulation of the classical problem of differentiation of Abelian functions over parameters. Its solution is relevant to a number of topical applications.

The talk presents a solution of this problem recently found by the speaker and D.Leykin.

A precise modern formulation of the problem involves the language of Differential Geometry. We obtain explicit expressions for the generators of the module of differentiations of a ring of Abelian functions. The families of curves, which we work with, are special deformations of the singularities $y^n - x^s$, where $\gcd(n, s) = 1$. Any algebraic curve has a bi-rationally equivalent model in such family. The choice of this type of families allows us to use methods and results of Singularity Theory, especially Arnold's convolution of invariants and the theorem of Zakalyukin on holomorphic vector fields tangent to the discriminant variety.

References

[1] V.M.Buchstaber, V.Z.Enolskii and D.V.Leykin,
Kleinian functions, hyperelliptic Jacobians and applications,
Reviews in Mathematics and Math. Physics,
v. 10, part 2, Gordon&Breach, London, 1997, 3-120.

[2] V.M.Buchstaber and D.V.Leykin,
Polynomial Lie algebras.
Funct. Anal. Appl. 36 (2002), no. 4, 267–280.

[3] V.M.Buchstaber and D.V.Leykin,
The heat equations in a nonholonomic frame.
Funct. Anal. Appl. 38 (2004), no. 2, 88–101.

[4] V.M.Buchstaber and D.V.Leykin,
Addition laws on Jacobian varieties of plane algebraic curves.
Proc. Steklov Math. Inst. 251 (2005), 49–120.

[5] V.M.Buchstaber and D.V.Leykin,
Differentiation of Abelian functions over parameters,
UMN, 62 (2007), no. 4, 153–154.

An *Abelian function* is, in the classical sense, a meromorphic function on a complex Abelian torus

$$T^g = \mathbb{C}^g/\Gamma,$$

where $\Gamma \subset \mathbb{C}^g$ is a rank $2g$ lattice.

That is f is Abelian iff

$$f(u) = f(u + \omega), \quad \text{for all } u \in \mathbb{C}^g \text{ and } \omega \in \Gamma.$$

Abelian functions form a differential field.

Complex dimension g of the torus is called the genus of a field.

A nonsingular algebraic curve C defines a lattice Γ as the set of all periods of basis holomorphic differentials: Let $\boldsymbol{\omega}$ be a vector of holomorphic differentials and γ be a cycle on C, then

$$\oint_{\gamma} \boldsymbol{\omega} \in \Gamma.$$

The resulting torus is called the Jacobian of C.

Suppose B is an open dense subset in $\mathbb{C}^d$.

We will consider a family V of nonsingular curves, depending linearly on a parameter $b \in B$. We use V to define over B a space of Jacobians U.

The space U is naturally fibred,

$$p : U \to B,$$

where the fiber over a point $b \in B$ is the Jacobian J_b of the curve with the parameter b.

The space U has a natural structure of smooth manifold. Consider on U the ring F of smooth fiber-wise Abelian functions.

Denote by F_b the restriction of F to a fiber J_b over $b \in B$.

Basic facts about Abelian functions on a Jacobian of genus g.

$\boxed{A_1}$ If $f \in F_b$, then $\partial_{u_i} f \in F_b$, $i = 1, \ldots, g$.

$\boxed{A_2}$ For any nonconstant $f_1, \ldots, f_{g+1}$ from F_b exists a polynomial P such that

$$P(f_1, \ldots, f_{g+1}) = 0, \quad \text{for all } u \in p^{-1}(b).$$

$\boxed{A_3}$ If $f \in F_b$ is any nonconstant function, then any $h \in F_b$ is a rational function of $(f, \partial_{u_1} f, \ldots, \partial_{u_g} f)$.

$\boxed{A_4}$ There exists an entire function $\vartheta : \mathbb{C}^g \longrightarrow \mathbb{C}$ such that

$$\partial_{u_i, u_j} \log \vartheta \in F_b, \quad i, j = 1, \ldots, g.$$

Problem:

Find the generators of the F-module $\mathrm{Der}(F)$ of derivations of the ring F.

As we show, to solve this problem it is sufficient

- Fix a family $V \longrightarrow B$ of curves of fixed genus g.

- Construct a basis $(\ell_1, \dots, \ell_d)$ of tangent bundle TB, acting on $f \in F$ as vector field along the base B.

- Construct operators $(H_1, \dots, H_d)$, , acting on $f \in F$ second order differential operators along fibers of $U \longrightarrow B$, such that sigma-function σ, associated with the family V, satisfies the system of equations

$$(\ell_i - H_i)\sigma = 0, \qquad i = 1, \dots, d.$$

Elliptic functions

Consider the family Weierstrass elliptic curves

$$V = \{(x, y, g_2, g_3) \in \mathbb{C}^2 \times B \mid y^2 = 4x^3 - g_2 x - g_3\}$$

where

$$B = \{(g_2, g_3) \in \mathbb{C}^2 \mid \Delta \neq 0\}, \quad \Delta = g_2^3 - 27g_3^2,$$

Here $g = 1$ and $d = 2$.
Vector fields

$$\ell_0 = 4g_2\partial_{g_2} + 6g_3\partial_{g_3}, \quad \ell_2 = 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3}$$

are tangent to B, since

$$\ell_0(\Delta) = 12\Delta, \quad \ell_2(\Delta) = 0.$$

A fiber of U is defined by the lattice

$$\oint_\gamma \omega, \qquad \omega = \frac{\mathrm{d}x}{y}.$$

The ring F is generated by g_2, g_3 and elliptic functions

$$\wp(u, g_2, g_3) \quad \text{and} \quad \wp'(u, g_2, g_3) = \partial_u \wp(u, g_2, g_3).$$

Obviously, the operator $L_1 = \partial_u$ is a derivation of F.

To find other generators of $\mathrm{Der}(F)$ we use the following properties of Weierstrass sigma-function $\sigma : \mathbb{C}^3 \to \mathbb{C}$.

(a) $\sigma(u, g_2, g_3)$ is entire in $(u, g_2, g_3) \in \mathbb{C}^3$.

(b) $\partial_u^2 \log(\sigma(u, g_2, g_3)) = -\wp(u, g_2, g_3) \in F$.

(c) $\sigma(u, g_2, g_3)$ is a solution of the system

$$Q_0(\sigma) = 0, \quad Q_2(\sigma) = 0, \quad \text{where}$$

$$\boxed{\begin{aligned} Q_0 &= 4g_2\partial_{g_2} + 6g_3\partial_{g_3} - u\partial_u + 1 \\ Q_2 &= 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3} - \frac{1}{2}\partial_u^2 - \frac{1}{24}g_2u^2 \end{aligned}}$$

Note: the operators have polynomial coefficients.

Now, using (b) and (c) we deduce the derivations of F.

We start from $Q_2(\sigma) = 0$,

$$Q_2 = \ell_2 - \frac{1}{2}\partial_u^2 - \frac{1}{24}g_2 u^2.$$

Divide $Q_2(\sigma) = 0$ by σ and rearrange the terms using

$$\zeta = \partial_u \log \sigma \quad \text{and} \quad \wp = -\partial_u^2 \log \sigma.$$

We obtain

$$\ell_2(\log(\sigma)) - \frac{1}{2}\zeta^2 + \frac{1}{2}\wp - \frac{1}{24}g_2 u^2 = 0.$$

Apply ∂_u^2, then, since $[\partial_u, \ell_2] = 0$, we have

$$-\ell_2(\wp) + \zeta\wp' - \wp^2 + \frac{1}{2}\wp'' - \frac{1}{12}g_2 = 0,$$

which means that

$$(\ell_2 - \zeta\partial_u)\wp \in F.$$

Thus, the operator $\boxed{L_2 = \ell_2 - \zeta\partial_u}$ is a derivation of F.

Similar calculation using the operator Q_0 leads to another derivation $\boxed{L_0 = \ell_0 - u\partial_u}$.

The generators of $\mathrm{Der}(F)$

$$\begin{aligned} L_0 &= 4g_2\partial_{g_2} + 6g_3\partial_{g_3} - u\partial_u, \\ L_1 &= \partial_u, \\ L_2 &= 6g_3\partial_{g_2} + \frac{1}{3}g_2^2\partial_{g_3} - \zeta(u, g_2, g_3)\partial_u, \end{aligned}$$

were found by F.G.Frobenius and L.Stickelberger*.

Note the structure relations

$$[L_0, L_k] = kL_k, \quad [L_1, L_2] = \wp(u, g_2, g_3)L_1.$$

B.A. Dubrovin† clarified the meaning of this result
for reconstructing the differential geometry
of the universal bundle of genus one Jacobians.
He named the connection on this bundle the *FS-connection*.

*Crelles Journal, Bd. 92. S. 311–337. (1882)

†*Geometry of 2D topological field theories*, Appendix C, Lect. Notes Math. 1620. (1994)

Weierstrass model of an algebraic curve

Let C be an irreducible algebraic curve of genus > 0 over $\mathbb{C}$.

Let p be a point in C. Consider the ring $M(p)$ of rational functions on C with their poles restricted to p only.
Construct the sequence $S(p) = (S_1, S_2, \dots)$, where $S_k = 1$ if a function with pole of order k exists in $M(p)$, else $S_k = 0$, $k \in \mathbb{N}$.

Definition. *If $S(p)$ is monotone p is a regular point. If $S(p)$ is non-monotone p is a Weierstrass point.*

Example.
If p is regular
$$S(p) = (0, 0, \dots, 0, 1, 1, 1 \dots).$$
If p is a *normal* Weierstrass point
$$S(p) = (0, 0, \dots, 0, 1, 0, 1, 1, \dots).$$
If p is a *hyperelliptic* Weierstrass point
$$S(p) = (0, 1, 0, 1, \dots, 0, 1, 1, \dots).$$

Let $p \in C$ be a Weierstrass point.
Denote by C_p the curve C punctured at p.

Consider the ring $\mathcal{O}$ of entire rational functions on C_p.

Definition. *Let $\phi \in \mathcal{O}$ be a nonconstant function. Total number of zeros of ϕ on C_p is called* order of ϕ *and denoted* $\mathrm{ord}\phi$*. Set* $\mathrm{ord}\phi = 0$*, when ϕ is a constant.*

If $\phi, \varphi \in \mathcal{O}$, then $\mathrm{ord}(\phi\varphi) = \mathrm{ord}\phi + \mathrm{ord}\varphi$.

Note, that $\phi \in \mathcal{O}$ extends uniquely to a rational function on C with a pole of order $\mathrm{ord}\phi$ at p.

Lemma. *Let $\phi, \varphi \in \mathcal{O}$ be nonconstant functions. Then φ is an entire algebraic function of ϕ.*

Let $x, y \in \mathcal{O}$ be nonconstant functions such that:
$n = \mathrm{ord} x$ is minimum;
$s = \mathrm{ord} y$ is minimum under the condition $\gcd(n, s) = 1$.
Assume, that y is normalized, so that

$$\mathrm{ord}(y^n - x^s) < ns.$$

Then, since y is an entire algebraic function of x, the function $y^n - x^s$ can be represented as a linear combination of terms $x^i y^j$ such that $\mathrm{ord}(x^i y^j) = ni + sj < ns$.

Thus, we come to the relation

$$y^n - x^s = \sum_{i,j \geq 0}^{q(i,j) > 0} \alpha_{i,j} x^i y^j,$$

where $q(i, j) = (n - j)(s - i) - ij$ and $\alpha_{i,j} \in \mathbb{C}$, which defines a Weierstrass model of the curve C.

The best known example of a Weierstrass model is the cubic equation for elliptic curves $(n = 2, s = 3)$

$$y^2 = 4x^3 - g_2 x - g_3.$$

Conclusion. Any irreducible algebraic curve has a Weierstrass model.

Note: for curves of genus > 1 Weierstrass model is not unique. Weierstrass models of a curve are bi-rationally equivalent.

Thus, following the idea of Weierstrass*, "generic curve" is replaced by the class of models parameterized by pairs (n, s), $s > n > 1$ and $\gcd(n, s) = 1$

$$V = \{(x, y; \lambda) \in \mathbb{C}^{2+d} \mid y^n = x^s + \sum_{i,j \geq 0}^{q(i,j)>0} \lambda_{q(i,j)} x^i y^j\},$$

where $d = \frac{(n+1)(s+1)}{2} - 1$ and $q(i, j) = (n-j)(s-i) - ij$.

Curves in Weierstrass' (n, s)-class are of genus *not greater than* $g = (n-1)(s-1)/2$.

For hyperelliptic curves of genus g we have $(n, s) = (2, 2g + 1)$.

*Abel'schen Funktionen. Ges. Werke, vol. 4. (1904)

Singularity Theory and (n, s)-curves.

Singularity Theory studies a function

$$f(x, y, \lambda) = y^n - x^s - \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{q(i,j)} x^i y^j,$$

as *miniversal unfolding* of *Pham singularity* $y^n - x^s$.

Miniversal unfolding has $(n-1)(s-1)$ parameters λ.

The number $m = \#\{\lambda_k \mid k < 0\}$ is the *modality* of f.

The discriminant variety $\Sigma \subset \mathbb{C}^{2g}$ of f:

$(\lambda \in \Sigma) \Leftrightarrow \big(\exists\, (x, y) \in \mathbb{C}^2 \colon f = f_x = f_y = 0$ at $(x, y, \lambda)\big)$.

Genus of a curve in Weierstrass' (n,s)-class
is $\leq\ (n-1)(s-1)/2$.

Genus of a miniversal unfolding, if $b \notin \Sigma$,
is $\geq\ (n-1)(s-1)/2$.

Impose the condition

$$\lambda_{q(i,j)} = 0, \quad \text{when } q(i,j) < 0,$$

on miniversal unfolding, or, equivalently,

$$\lambda_{q(s-1,j)} = \lambda_{q(i,n-1)} = 0$$

on Weierstrass model.

We obtain a family of curves over $B = \mathbb{C}^{2g-m} \cap (\mathbb{C}^{2g} \backslash \Sigma)$ of constant genus $\boxed{g = (n-1)(s-1)/2}$.

The class (n,s)-curves is the intersection
of the classes of miniversal unfoldings and
Weierstrass' models.

A family V of (n,s)-curves is defined with the help of polynomial

$$f(x,y,\lambda) = y^n - x^s - \sum_{\substack{0 \le i < s-1 \\ 0 \le j < n-1}}^{q(i,j)>0} \lambda_{q(i,j)} x^i y^j,$$

where $q(i,j) = (n-j)(s-i) - ij$,

$$V = \{(x,y;\lambda) \in \mathbb{C}^{2+d} \mid f(x,y,\lambda) = 0\},$$

where $\quad d = 2g - m.$

- Newton polygon of f is a triangle.

- f is homogeneous with respect to the grading:

$$\boxed{\deg x = n, \quad \deg y = s, \quad \deg \lambda_k = k.}$$

- a generic curve from V, considered as n-fold covering of S^2, has $(n-1)s + 1 = 2g + n$ branch points:
  the point at infinity has branch number $n-1$;
  the rest $2g + n - 1$ points have branch number 1, by Riemann-Hurwitz formula.

Discriminant of (n,s)-curve and tangent fields

In what follows we use an obvious renumbering of λ_k.

Let $\Sigma \subset \mathbb{C}^{2g-m}$ be the discriminant of $f(x,y,\lambda)$.

We apply a theorem due to V.M. Zakalyukin* to obtain holomorphic vector fields tangent to Σ:

$$\mathcal{L} = (\ell_1, \ldots, \ell_{2g})^t = T(\lambda)(\underbrace{0, \ldots, 0}_{m}, \partial_{\lambda_1}, \ldots, \partial_{\lambda_{2g-m}})^t.$$

Lemma. *Matrix $T(\lambda)$ is polynomial in λ*

Let $\Delta(\lambda) = \det T(\lambda)$. Then

$$\Sigma = \{\lambda \in \mathbb{C}^{2g-m} \mid \Delta(\lambda) = 0\}.$$

Vector field ℓ_i is tangent to Σ, that is

$$\ell_i(\Delta(\lambda)) = \phi_i(\lambda)\Delta(\lambda), \quad \text{where} \quad \phi_i(\lambda) \in \mathbb{C}[\lambda].$$

for $i = 1, \ldots, 2g$.

*Funct. Anal. Appl. 10 (1976), no. 2, 139–140.

Theorem. *Polynomial frame* $\mathcal{L} = (\ell_1, \ldots, \ell_{2g})^t$ *is the* $2g$*-dimensional basis of* $\mathbb{C}[\lambda]$*-module of global sections of* $(2g - m)$*-dimensional bundle* TB*.*

Moreover, in

$$[\ell_i, \ell_j] = \sum_{h=1}^{2g} c_{ij}^h(\lambda)\ell_h, \qquad i, j = 1, \ldots, 2g,$$

the structure functions $c_{ij}^h(\lambda)$ *are polynomial in* λ*.*

The frame $\mathcal{L}$ can be modified by taking linear combinations of fields ℓ_i with $\mathbb{C}[\lambda]$ coefficients, so that the matrix $T(\lambda) = (T_{ij}(\lambda))$ becomes symmetric.

Then $T(\lambda)$ is the matrix of Arnold's convolution:

$$\lambda_i * \lambda_j = \ell_i\lambda_j = \ell_j\lambda_i = T_{ij}(\lambda).$$

We assume in what follows that $\mathcal{L}$ is the symmetrized frame.

Gauß-Manin (GM) connection on the bundle of (n, s)-curves punctured at (∞).

The equation $f(x, y, \lambda) = 0$ in $\mathbb{C}^{2+2g-m}$ defines the family V of (n, s)-curves over $B = \mathbb{C}^{2g-m} \backslash \Sigma$.

Consider the bundle $\overset{\circ}{p} : \overset{\circ}{V} \longrightarrow B$ whose fiber is the curve

$$\overset{\circ}{V}_b = \{(x, y) \in \mathbb{C}^2 \mid f(x, y, b) = 0\}$$

with a puncture at infinity.

Let $H^1(\overset{\circ}{V}_b, \mathbb{C})$ be the linear $2g$-dimensional vector space of holomorphic 1-forms on $\overset{\circ}{V}_b$.

Consider associated with $\overset{\circ}{p} : \overset{\circ}{V} \longrightarrow B$ locally trivial vector bundle $\varpi : \Omega^1 \longrightarrow B$ whose fiber is $H^1(\overset{\circ}{V}_b, \mathbb{C})$.

A connection in Ω^1 is GM connection on $\overset{\circ}{V}$.

Since (∞) belongs to all curves from V, we can construct a global section of Ω^1 by taking the classical basis of Abelian differentials of first and second kind.

Let $D(x,y,\lambda)$ be the vector

$$D(x,y,\lambda) = (D_1(x,y,\lambda),\ldots,D_{2g}(x,y,\lambda))$$

of canonical basis 1-forms from $H^1(\overset{\circ}{V}_b,\mathbb{C})$.

Its matrix of periods Ω satisfies the Legendre relation*

$$\Omega^t J\Omega = 2\pi\imath J, \qquad \text{where} \quad J = \begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix}.$$

Classical theory of Abelian differentials asserts that such basis exists and provides a means to construct it†.

*The particular case of Riemann-Hodge relations.

†H.F.Baker, Abelian Functions, CUP, 1997

The Christoffel coefficient of GM connection

$$\Gamma_j = (\Gamma^k_{j,i}), \quad i, j, k = 1, \ldots, 2g,$$

associated to the field ℓ_j is uniquely defined by the relation

The holomorphic vector-valued 1-form
$\ell_j(D(x,y,\lambda)) + \Gamma_j \, D(x,y,\lambda)$
is exact 'on the curve'.

Here 'on the curve' implies that $f(x,y,\lambda) = 0$.

Integrating over basis cycles on a fiber we obtain

$$\ell_j(\Omega) + \Gamma_j \, \Omega = 0.$$

From Legendre identity follows $\det \Omega \neq 0$.

Corollary. $\Gamma_j = -\ell_j(\Omega)\Omega^{-1}$.

Applying ℓ_j to Legendre identity we come to

Corollary. $J\Gamma_i + \Gamma_i^t J = 0$.

Sigma-function

Def. *Covariant shift*

$$W_{a,b,c}(f(u)) := \exp\{\pi\imath\,(\langle 2u+a,b\rangle + c)\}f(u+a),$$

where $u, a, b \in \mathbb{C}^g$; $c \in \mathbb{C}/2\mathbb{Z}$, $\imath^2 = -1$
and $\langle\,\cdot\,,\,\cdot\,\rangle$ is Euclidean scalar product.

Group of covariant shifts $:= Sh$

$$W_{a_2,b_2,c_2}W_{a_1,b_1,c_1} = W_{a_1+a_2,\ b_1+b_2,\ c_1+c_2+\langle b_1,a_2\rangle-\langle a_1,b_2\rangle}$$

We use representations $\mathbb{Z}^g \times \mathbb{Z}^g \to Sh$ defined by the formula

$$(n, n') \;\mapsto\; (a, b, c) = ((n, n')\Omega,\ \phi(n, n')).$$

Where:

(1) $\Omega \in \mathrm{Sp}(2g, \mathbb{C})$: $\Omega^t J \Omega = J$,

(2) $\phi : \mathbb{Z}^{2g} \to \mathbb{Z}_2$ is an Arf function:
for all N_1 and N_2 in $\mathbb{Z}^{2g}$ *Arf identity* holds

$$\phi(N_1 + N_2) = \phi(N_1) + \phi(N_2) + N_1 J N_2^t \mod 2;$$

To define an Arf function ϕ fix $(\varepsilon, \varepsilon') \in \mathbb{Z}^{2g}$ then

$$\phi(n, n') = \langle n + \varepsilon, n' + \varepsilon' \rangle - \langle \varepsilon, \varepsilon' \rangle \mod 2.$$

Write Ω in block form $\Omega = \begin{pmatrix} \Omega_{1,1} & \Omega_{1,2} \\ \Omega_{2,1} & \Omega_{2,2} \end{pmatrix}$;
then we have a representation:

$$W_\Omega^{\varepsilon,\varepsilon'}(n, n') := W_{a,b,c},$$

where

$$a = (n\Omega_{1,1} + n'\Omega_{2,1})$$
$$b = (n\Omega_{1,2} + n'\Omega_{2,2})$$
$$c = \langle n + \varepsilon, n' + \varepsilon' \rangle - \langle \varepsilon, \varepsilon' \rangle$$

Let $|\Omega_{1,1}| \neq 0$. Set

$$G_\Omega(u) = |\Omega_{1,1}|^{-1/2} \exp\Big\{-\frac{\pi\imath}{2}\, u\varkappa u^t\Big\}, \qquad \varkappa = \Omega_{1,1}^{-1}\Omega_{1,2}.$$

Def. $\boxed{\sigma(u,\Omega;\varepsilon,\varepsilon') := \sum_{(n,n')\in\mathbb{Z}^{2g}} W_\Omega^{\varepsilon,\varepsilon'}(n,n')\, G_\Omega(u)}$

Theorem. $\sigma(u,\Omega;\varepsilon,\varepsilon')$ is entire function of $u \in \mathbb{C}^g$ iff

$\boxed{\mathrm{Im}\,\tau \quad \text{is positive definite.}}$

where

$$\tau = \Omega_{2,1}\Omega_{1,1}^{-1}, \quad \Rightarrow \quad \Omega_{2,2} = \tau\Omega_{1,1}\varkappa + (\Omega_{1,1}^t)^{-1}$$

Linear operators that annihilate sigma-function

Let $\alpha_j = (\alpha_j^{kl})$, $\quad \beta_j = (\beta_{jk}^l)$, $\quad$ and $\quad \gamma_j = (\gamma_{jkl})$,
where $k, l = 1, \ldots, g$ and $j = 1, \ldots, 2g$.
Assign $\begin{pmatrix} \alpha_j & (\beta_j)^t \\ \beta_j & \gamma_j \end{pmatrix} = -J\Gamma_j$, where Γ_j is the Christoffel coefficient of GM connection, and set

$$\begin{aligned} H_j =& \frac{1}{2}\alpha_j^{kl}(\lambda)\partial_{u_k}\partial_{u_l} + \beta_{jk}^l(\lambda)u_k\partial_{u_l} + \frac{1}{2}\gamma_{jkl}(\lambda)u_k u_l, \\ & + \frac{1}{8}\ell_j(\log\Delta(\lambda)) + \frac{1}{2}\beta_{jk}^k(\lambda), \end{aligned}$$

where the summation from 1 to g extends over the repeated indices.

Theorem. *Sigma-function $\sigma(u, \lambda)$ satisfies $2g$ linear differential equations*

$$(\ell_j - H_j)\sigma(u, \lambda) = 0, \qquad i = 1, \ldots, 2g.$$

Properties of sigma-function.*

(a) $\sigma(u,\lambda)$ is an entire function of $u \in \mathbb{C}^g$ and $\lambda \in \mathbb{C}^d$.

(b) $\partial_{u_i,u_j} \log(\sigma(u,\lambda)) = -\wp_{ij}(u,\lambda) \in F$
whenever $\lambda \in B$, $i,j = 1,\dots,g$.

(c) $\sigma(u,\lambda)$ is a solution of the system

$$(\ell_j - H_j)\sigma(u,\lambda) = 0, \quad j = 1,\dots,2g.$$

Solution of the problem.

Derivations of F.†

Theorem. *F-module* $\mathrm{Der}(F)$ *has* $3g$ *generators*

$$L_i = \partial_{u_i},$$
$$L_{j+g} = \ell_j - (\alpha_j^{kl}\zeta_k(u,\lambda) + \beta_{jk}^l u_k)\partial_{u_l},$$
$$i = 1,\dots,g; \qquad j = 1,\dots,2g.$$

where $\zeta_k(u,\lambda) = \partial_{u_k} \log \sigma(u,\lambda)$.

---

*Funct. Anal. Appl. 38 (2004), no. 2, 88–101.

†UMN, 62 (2007), no. 4, 153–154.

g=2. Basis vector fields $\{\ell_j\}$.

The symmetric matrix T, which transforms the standard fields $\partial_{\lambda_4}, \partial_{\lambda_6}, \partial_{\lambda_8}, \partial_{\lambda_{10}}$ to the basis fields $\ell_0, \ell_2, \ell_4, \ell_6$,

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ * & \frac{40\lambda_8 - 12\lambda_4^2}{5} & \frac{50\lambda_{10} - 8\lambda_4\lambda_6}{5} & -\frac{4\lambda_4\lambda_8}{5} \\ * & * & \frac{20\lambda_4\lambda_8 - 12\lambda_6^2}{5} & \frac{30\lambda_4\lambda_{10} - 6\lambda_6\lambda_8}{5} \\ * & * & * & \frac{4\lambda_6\lambda_{10} - 8\lambda_8^2}{5} \end{pmatrix}$$

$$[\ell_0, \ell_k] = k\ell_k, \quad k = 2, 4, 6;$$
$$[\ell_2, \ell_4] = 2\ell_6 - \frac{8}{5}\lambda_4\ell_2 + \frac{8}{5}\lambda_6\ell_0;$$
$$[\ell_2, \ell_6] = -\frac{4}{5}\lambda_4\ell_4 + \frac{4}{5}\lambda_8\ell_0;$$
$$[\ell_4, \ell_6] = 2\lambda_4\ell_6 - \frac{6}{5}\lambda_6\ell_4 + \frac{6}{5}\lambda_8\ell_2 - 2\lambda_{10}\ell_0;$$

The vector fields are homogeneous $\deg \ell_j = j$

$g = 2$. The operators $\{H_j\}$.

The operators are homogeneous $\deg H_j = j$;
however $\deg u_i = -i$.

$$H_0 = u_1\partial_{u_1} + 3u_3\partial_{u_3} - 3$$

$$10H_2 = 5\partial_{u_1}^2 + 10u_1\partial_{u_3} - 8\lambda_4 u_3\partial_{u_1} - 3\lambda_4 u_1^2 + (15\lambda_8 - 4\lambda_4^2)u_3^2$$

$$5H_4 = 5\partial_{u_1}\partial_{u_3} + 5\lambda_4 u_3\partial_{u_3} - 6\lambda_6 u_3\partial_{u_1} - 5\lambda_4 - \lambda_6 u_1^2 + 5\lambda_8 u_1 u_3 + 3(5\lambda_{10} - \lambda_4\lambda_6)u_3^2$$

$$10H_6 = 5\partial_{u_3}^2 - 6\lambda_8 u_3\partial_{u_1} - 5\lambda_6 - \lambda_8 u_1^2 + 20\lambda_{10} u_1 u_3 - 3\lambda_4\lambda_8 u_3^2$$